

AROUND POISSON–MEHLER SUMMATION FORMULA

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ABSTRACT. We study some simple generalization of the Poisson–Mehler summation formula (1.1). Namely we exploit farther, the recently obtained equality $\gamma_{m,n}(x, y|t, q) = \gamma_{0,0}(x, y|t, q) Q_{m,n}(x, y|t, q)$ where $\gamma_{m,n}(x, y|t, q) = \sum_{i \geq 0} \frac{t^i}{[i]_q!} H_{i+n}(x|q) H_{m+i}(y|q)$, $\{H_n(x|q)\}_{n \geq -1}$ are the so called q –Hermite polynomials (qH) and $\{Q_{m,n}(x, y|t, q)\}_{n,m \geq 0}$ are certain polynomials in x, y of order $m+n$ that are also rational functions in t and q . We study properties of polynomials $Q_{m,n}(x, y|t, q)$ expressing them with the help the so called Al-Salam–Chihara (ASC) polynomials and using them in expansion of the reciprocal of the right hand side of the Poisson–Mehler formula. We also point out the fact that spaces $\text{span}\{Q_{i,n-i}(x, y|t, q) : i = 0, \dots, n\}_{n \geq 0}$ are orthogonal with respect to certain two dimensional measure (two dimensional (t, q) –Normal distribution) on the square $\{(x, y) : |x|, |y| \leq 2/\sqrt{1-q}\}$.

1. INTRODUCTION AND AUXILIARY RESULTS

1.1. **Preface.** As stated in the abstract we consider various generalizations of the celebrated Poisson–Mehler formula:

$$(1.1) \quad \sum_{n \geq 0} \frac{\rho^n}{[n]_q!} H_n(x|q) H_n(y|q) = \frac{(\rho^2)_\infty}{\prod_{j=0}^\infty \omega(x\sqrt{1-q}/2, y\sqrt{1-q}/2|\rho q^j)}.$$

where $\{H_n\}_{n \geq 0}$ denote q –Hermite polynomials and $\omega(x, y|t)$ are certain polynomials symmetric in x and y of order two. They are defined by (1.18). There exist many proofs of (1.1) (e.g. see [13], [14], [1], [8]). Recently in [9] certain generalization of (1.1) has been proved. It was used in calculating moments of the so called Askey–Wilson distribution. The generalization is given by (2.4), below with functions γ defined by (2.1) which form generalizations of the left hand side of (1.1). In these generalizations there appears a family of polynomials of two variables $Q_{n,m}(x, y|\rho, q)$ of order $m+n$. We play around with these polynomials finding their generating function, expressing them with the help of other families of polynomials, using them in an expansion of the reciprocal of the right hand side of (1.1) in an infinite series. This time expansion is symmetric in x and y as compared to the non-symmetric (for each finite sum) expansion of the same function presented in [8](formula 5.3). We also analyze two dimensional measure (so called $(\rho, q) - 2N$ measure) on the square $S(q) \times S(q)$ with the density defined by (2.7), where $S(q)$ is defined by (1.3) and point out the rôle of the polynomials $Q_{n,m}$ in analysis of

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this measure. In particular we introduce spaces of functions of two variables

$$(1.2) \quad \Lambda_n(x, y | \rho, q) = \text{span} \{Q_{i, n-i}(x, y, | \rho, q), i = 0, \dots, n\}, n \geq 0$$

and show that they are orthogonal with respect to this $(\rho, q) - 2N$ measure. Hence these spaces form the direct sum decomposition of the space of functions that are square integrable with respect to $(\rho, q) - 2N$ measure.

The paper is organized as follows. In the next two subsections we provide simple introduction to q -series theory presenting typical notation used and presenting a few typical families of the so called basic orthogonal polynomials. The word basic comes from the base which is the parameter in most cases denoted by q . Then in Section 2 we present our main results, open questions and remarks are in Section 3 while less interesting laborious proofs are in Section 4.

1.2. Notation. We use notation traditionally used in the so called q -series theory. Since not all readers are familiar with it we will recall now this notation.

q is a parameter. We will assume that $-1 < q \leq 1$ unless otherwise stated. Let us define $[0]_q = 0$; $[n]_q = 1 + q + \dots + q^{n-1}$, $[n]_q! = \prod_{j=1}^n [j]_q$, with $[0]_q! = 1$ and

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \frac{[n]_q!}{[n-k]_q! [k]_q!} & , \quad n \geq k \geq 0 \\ 0 & , \quad \text{otherwise} \end{cases}.$$

It will be useful to use the so called q -Pochhammer symbol for $n \geq 1$:

$$(a; q)_n = \prod_{j=0}^{n-1} (1 - aq^j),$$

$$(a_1, a_2, \dots, a_k; q)_n = \prod_{j=1}^k (a_j; q)_n.$$

with $(a; q)_0 = 1$. Often $(a; q)_n$ as well as $(a_1, a_2, \dots, a_k; q)_n$ will be abbreviated to $(a)_n$ and $(a_1, a_2, \dots, a_k)_n$, if it will not cause misunderstanding.

It is easy to notice that $(q)_n = (1 - q)^n [n]_q!$ and that

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \frac{(q)_n}{(q)_{n-k} (q)_k} & , \quad n \geq k \geq 0 \\ 0 & , \quad \text{otherwise} \end{cases}.$$

Notice that $[n]_1 = n$, $[n]_1! = n!$, $\begin{bmatrix} n \\ k \end{bmatrix}_1 = \binom{n}{k}$, $(a; 1)_n = (1 - a)^n$ and $[n]_0 = \begin{cases} 1 & \text{if } n \geq 1 \\ 0 & \text{if } n = 0 \end{cases}$, $[n]_0! = 1$, $\begin{bmatrix} n \\ k \end{bmatrix}_0 = 1$, $(a; 0)_n = \begin{cases} 1 & \text{if } n = 0 \\ 1 - a & \text{if } n \geq 1 \end{cases}$.

In the sequel we shall also use the following useful notation:

$$(1.3) \quad S(q) = \begin{cases} [-\frac{2}{\sqrt{1-q}}, \frac{2}{\sqrt{1-q}}]_{\mathbb{R}} & \text{if } |q| < 1 \\ \mathbb{R} & \text{if } q = 1 \end{cases},$$

$$(1.4) \quad I_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}.$$

1.3. Polynomials.

1.3.1. q -Hermite. Let $\{H_n(x|q)\}_{n \geq 0}$ denote the family of the so called q -Hermite (briefly qH) polynomials. That is the one parameter family of orthogonal polynomials satisfying the following 3-term recurrence:

$$(1.5) \quad H_{n+1}(x|q) = xH_n(x|q) - [n]_q H_{n-1}(x|q),$$

with $H_{-1}(x|q) = 0$ and $H_0(x|q) = 1$. In fact in the literature (see e.g. [14], [13], [12]) function more often re-scaled versions of these polynomials. Namely more often appear under the name of q -Hermite polynomials the following polynomials $\{h_n(x|q)\}_{n \geq 0}$ defined by their 3-term recurrence:

$$(1.6) \quad h_{n+1}(x|q) = 2xh_n(x|q) - (1 - q^n)h_{n-1}(x|q),$$

with $h_{-1}(x|q) = 0$ and $h_0(x|q) = 1$. These polynomials are related to one another by the relationship $\forall n \geq -1$:

$$H_n(x|q) = \frac{h_n(x\sqrt{1-q}/2|q)}{(1-q)^{n/2}},$$

for $|q| < 1$. For $q = 1$ $h_n(x|1) = 2^n x^n$ while $H_n(x|1) = H_n(x)$, where polynomials $H_n(x)$ are the so called probabilistic Hermite polynomials i.e. monic¹ polynomials orthogonal with respect to $\exp(-x^2/2)$. Observe further that $h_n(x|0) = U_n(x)$ where U_n denotes the so called Chebyshev polynomial of the second kind (for details see e.g. [14]).

The polynomials H_n have a nice probabilistic interpretation (see e.g. [9]) and besides constitute really the generalization of the ordinary Hermite polynomials. That is why we will use them in this paper. The results presented here can be easily adopted and expressed in terms of polynomials h_n .

Their generating function is given by the following formula

$$(1.7) \quad \varphi_H(x|\rho, q) = \sum_{n \geq 0} \frac{\rho^n}{[n]_q!} H_n(x|q) = \frac{1}{\prod_{i=0}^{\infty} v(x\sqrt{1-q}/2|\rho\sqrt{1-qq^i})},$$

convergent for $|\rho(1-q)| < 1$, $x \in S(q)$, where we denoted

$$(1.8) \quad v(x|t) = 1 - 2xt + t^2.$$

Let us observe that $\forall y \in [-1, 1]$, $t \in \mathbb{R} : v(y|t) \geq 0$.

Besides we have:

$$(1.9) \quad \int_{S(q)} H_n(x|q) H_m(x|q) f_N(x|q) dx = [n]_q! \delta_{mn},$$

with

$$(1.10) \quad f_N(x|q) = \frac{\sqrt{(1-q)(4-(1-q)x^2)}(q)_{\infty}}{2\pi} \prod_{i=1}^{\infty} l(x\sqrt{1-q}/2|q^i).$$

for $x \in S(q)$, where

$$(1.11) \quad l(x|a) = (1+a)^2 - 4ax^2.$$

One shows that

$$(1.12) \quad \lim_{q \rightarrow 1^-} f_N(x|q) = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2),$$

$$(1.13) \quad \lim_{q \rightarrow 1^-} \frac{1}{\prod_{i=0}^{\infty} v(x|\rho q^i)} = \exp(xt - x^2/2).$$

¹i.e. polynomials with leading coefficient equal to 1.

Apart from q -Hermite polynomials we will need the so called big q -Hermite (briefly bqH) polynomials $\{H_n(x|a, q)\}_{n \geq -1}$. They are defined through their 3-term recurrence:

$$(1.14) \quad H_{n+1}(x|a, q) = (x - aq^n)H_n(x|a, q) - [n]_q H_{n-1}(x|a, q),$$

with $H_{-1}(x|a, q) = 0$, $H_0(x|a, q) = 1$. To support intuition let us remark that $H_n(x|a, 1) = H_n(x - a)$ and $H_n(x|a, 0) = U_n(x/2) - aU_{n-1}(x/2)$.

One knows its relationship with the q -Hermite polynomials:

$$H_n(x|a, q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q (-a)^k q^{\binom{k}{2}} H_{n-k}(x|q),$$

and that (see e.g. [12] with an obvious modification for polynomials H_n):

$$\begin{aligned} \int_{S(q)} H_n(x|a, q) H_m(x|a, q) f_{bN}(x|a, q) dx &= [n]_q! \delta_{mn}, \\ \sum_{n \geq 0} \frac{t^n}{[n]_q!} H_n(x|a, q) &= \varphi_H(x|t, q) ((1-q)at)_\infty \end{aligned}$$

where

$$f_{bN}(x|a, q) = f_N(x|q) \varphi_H(x|a, q)$$

We will need the following Lemma concerning another relationship between polynomials $H_n(x|q)$ and $H_n(x|a, q)$.

Lemma 1. *i) Let us define for $\forall n \geq 0; x \in S(q); (1-q)t^2 < 1$: $\eta_n(x|t, q) = \sum_{i \geq 0} \frac{t^i}{[i]_q!} H_{i+n}(x|q)$. Then*

$$\eta_n(x|t, q) = H_n(x|t, q) \eta_0(x|t, q),$$

where $H_n(x|t, q)$ is the bqH polynomial defined by (1.14).

Proof. Is shifted to section 4 □

1.3.2. Al-Salam-Chihara. Next family of polynomials that we are going to consider depends on 2 (apart from q) parameters denoted by a and b , that satisfy the following 3-term recurrence (see e.g. [12]):

$$(1.15) \quad A_{n+1}(x|a, b, q) = (2x - (a+b)q^n)A_n(x|a, b, q) - (1 - abq^{n-1})(1 - q^n)A_{n-1}(x|y, \rho, q),$$

with $A_{-1}(x|a, b, q) = 0$, $A_0(x|a, b, q) = 1$. These polynomials will be called Al-Salam-Chihara polynomials $\{A_n(x|a, b, q)\}_{n \geq -1}$ (briefly ASC). It follows from Favard's theorem that the measure that makes these polynomials orthogonal is positive if $\forall n \geq 1 : (1 - abq^{n-1})(1 - q^n) \geq 0$, which for $|q| \leq 1$ is reduced to $|ab| \leq 1$.

In the sequel in fact we will consider these polynomials for complex parameters forming a conjugate pair and also re-scaled. Namely we will take $a = \frac{\sqrt{1-q}}{2} \rho(y - i\sqrt{\frac{4}{1-q} - y^2})$, $b = \frac{\sqrt{1-q}}{2} \rho(y + i\sqrt{\frac{4}{1-q} - y^2})$, with $y \in S(q)$ and $|\rho| < 1$. More precisely we will consider polynomials $\{P_n(x|y, \rho, q)\}_{n \geq 0}$ defined by:

$$A_n\left(x \frac{\sqrt{1-q}}{2} |a, b, q\right) / (1-q)^{n/2} = P_n(x|y, \rho, q).$$

One can easily notice that $a + b = \rho y \sqrt{1 - q}$, $ab = \rho^2$ and thus that the polynomials P_n satisfy the following 3-term recurrence:

$$(1.16) \quad P_{n+1}(x|y, \rho, q) = (x - \rho y q^n) P_n(x|y, \rho, q) - [n]_q (1 - \rho^2 q^{n-1}) P_{n-1}(x|y, \rho, q),$$

with $P_{-1}(x|y, \rho, q) = 0$, $P_0(x|y, \rho, q) = 1$.

Remark 1. To support intuition let us remark following e.g. [9] that $P_n(x|y, \rho, 1) = H_n\left(\frac{x - \rho y}{\sqrt{1 - \rho^2}}\right) (1 - \rho^2)^{n/2}$. On the other hand $P_n(x|y, \rho, 0) = U_n(x/2) - \rho y U_{n-1}(x/2) + \rho^2 U_{n-2}(x/2)$, where $U_n(x)$ denotes Chebyshev polynomial of the second kind.

It is known see e.g. [13], [11] or [9] that the polynomials P_n have the following generating function:

$$\varphi_P(x|y, \rho, t, q) = \sum_{n \geq 0} \frac{t^n}{[n]_q!} P_n(x|y, \rho, q) = \prod_{j=0}^{\infty} \frac{v(y\sqrt{1-q}/2 | \rho t \sqrt{1-qq^j})}{v(x\sqrt{1-q}/2 | t\sqrt{1-qq^j})}$$

convergent for $|t\sqrt{1-q}|, |\rho| < 1$, $x, y \in S(q)$.

We also have (see e.g. [9]) or :

$$(1.17) \quad \int_{-1}^1 P_n(x|y, \rho, q) P_m(x|y, \rho, q) f_{CN}(x|y, \rho, q) = \delta_{nm} [n]_q! (\rho^2)_n,$$

where

$$f_{CN}(x|y, \rho, q) = f_N(x|q) \frac{(\rho^2)_{\infty}}{\prod_{i=0}^{\infty} \omega(x\sqrt{1-q}/2, y\sqrt{1-q}/2 | \rho q^i)},$$

with

$$(1.18) \quad \omega(x, y | \rho) = (1 + \rho^2)^2 - 4\rho(1 + \rho^2)xy + 4\rho^2(x^2 + y^2).$$

We will call the densities f_N and f_{CN} respectively q -Normal and (q, ρ) -Conditional Normal. The names are justified by the nice probabilistic interpretations of these densities presented e.g. in [4], [2], [3], [11], [9]. Besides apart from (1.12) we also have:

$$\lim_{q \rightarrow 1^-} f_{CN}(x|y, \rho, q) = \exp\left(\frac{(x - \rho y)^2}{2(1 - \rho^2)}\right) / \sqrt{2\pi(1 - \rho^2)}.$$

1.3.3. General Result. We end up this section by presenting of an auxiliary simple result that will be used in following sections many times.

Proposition 1. *Let $\sigma_n(\rho|q) = \sum_{i \geq 0} \frac{\rho^i}{[i]_q!} \xi_{n+i}$ for $|\rho| < 1$, $-1 < q \leq 1$ and certain sequence $\{\xi_m\}_{m \geq 0}$ such that σ_n exists for every n . Then*

$$(1.19) \quad \sigma_n(\rho q^m | q) = \sum_{k=0}^m (-1)^k \begin{bmatrix} m \\ k \end{bmatrix}_q q^{\binom{k}{2}} (1 - q)^k \rho^k \sigma_{n+k}(\rho | q).$$

Proof. An easy, not very interesting proof by induction is shifted to section 4. \square

2. MAIN RESULTS

Our main interest in this paper are the generalizations of the famous Poisson-Mehler formula i.e. formula given by (1.1).

Convergence in (1.1) takes place for $x, y \in S(q)$, $|\rho| < 1$ and for $|q| < 1$ is uniform. For $q = 1$ we have almost uniform convergence.

Let us define the following function:

$$(2.1) \quad \gamma_{i,j}(x, y|\rho, q) = \sum_{n \geq 0} \frac{\rho^n}{[n]_q!} H_{n+i}(x|q) H_{n+j}(y|q).$$

As a immediate corollary of Proposition 1 we have:

Corollary 1. *For $|q| < 1$ we have:*

$$(2.2) \quad \gamma_{i,j}(x, y|\rho q^m, q) = \sum_{k=0}^m (-1)^k \begin{bmatrix} m \\ k \end{bmatrix}_q q^{\binom{k}{2}} (1-q)^k \rho^k \gamma_{i+k, j+k}(x, y|\rho, q),$$

$$(2.3) \quad H_i(x|q) H_j(y|q) = \sum_{k \geq 0} (-1)^k q^{\binom{k}{2}} \frac{\rho^k}{(q)_k} \gamma_{i+k, j+k}(x, y|\rho, q).$$

Proof. First assertion we get by apply directly (1.19) by setting $\sigma_n = \gamma_{i,j}$. Second assertion we get noticing firstly that $\lim_{m \rightarrow \infty} \begin{bmatrix} m \\ k \end{bmatrix}_q = \frac{1}{[k]_q!}$ and then that $\frac{(1-q)^k}{[k]_q!} = \frac{1}{(q)_k}$. \square

It was shown in [8] (Lemma 3) that:

$$(2.4) \quad \gamma_{i,j}(x, y|\rho, q) = Q_{i,j}(x, y|\rho, q) \gamma_{0,0}(x, y|\rho, q),$$

where $Q_{i,j}(x, y|\rho, q)$ is a certain polynomial in x, y of order $i+j$. Notice that (2.4) can be viewed as a generalization of (1.1).

Moreover it was shown there that

$$(2.5) \quad Q_{i,j}(x, y|\rho, q) = \sum_{s=0}^j (-1)^s q^{\binom{s}{2}} \begin{bmatrix} j \\ s \end{bmatrix}_q \rho^s H_{j-s}(y|q) P_{i+s}(x|y, \rho, q) / (\rho^2)_{i+s},$$

and $Q_{i,j}(x, y|\rho, q) = Q_{j,i}(y, x|\rho, q)$.

In particular we have

$$(2.6) \quad Q_{i,0}(x, y|\rho, q) = P_i(x|y, \rho, q) / (\rho^2)_i.$$

To analyze further properties of polynomials $Q_{i,j}$ let us introduce the following 2 dimensional density defined for $S^2(q) \stackrel{df}{=} S(q) \times S(q)$.

$$(2.7) \quad f_{2D}(x, y|\rho, q) = f_{CN}(x|y, \rho, q) f_N(y|q).$$

Measure that has density f_{2D} will be called (ρ, q) -bivariate Normal (briefly (ρ, q) -2N). Obviously $f_{2D}(x, y|\rho, q) = \gamma_{0,0}(x, y|\rho, q) f_N(x|q) f_N(y|q)$. Its applications in probability and stochastic processes have been presented in [6] and [7].

Here below we give another interpretation of polynomials $Q_{n,m}$ as well as its connection with so called big q -Hermite polynomials.

Proposition 2. *We have i) $\forall i, j, m, k, i+j \neq m+k$:*

$$\int_{S^2(q)} Q_{i,j}(x, y|\rho, q) Q_{m,k}(x, y|\rho, q) f_{2D}(x, y|\rho, q) dx dy = 0.$$

ii)

$$\sum_{n,m \geq 0} \frac{t^n s^m}{[n]_q! [m]_q!} Q_{n,m}(x, y|\rho, q) = \varphi_H(x|t, q) \varphi_H(y|s, q) \sum_{k \geq 0} \frac{\rho^k}{[k]_q!} H_k(x|t, q) H_k(y|s, q),$$

where function φ_H is defined by (1.7).

iii) $\forall m \geq 0$:

$$(2.8) \quad Q_{i,j}(x, y, \rho q^m, q) \prod_{i=0}^{m-1} \omega\left(x\sqrt{1-q}/2, y\sqrt{1-q}/2|\rho q^i\right) =$$

$$(2.9) \quad (\rho^2)_{2m} \sum_{k=0}^m (-1)^k \begin{bmatrix} m \\ k \end{bmatrix}_q q^{\binom{k}{2}} (1-q)^k \rho^k Q_{i+k, j+k}(x, y|\rho, q),$$

where polynomial ω is defined by (1.18). In particular we have:

$$(2.10) \quad \prod_{i=0}^{n-1} \omega\left(x\sqrt{1-q}/2, y\sqrt{1-q}/2|\rho q^i\right) = (\rho^2)_{2n} \sum_{k=0}^n (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}} (1-q)^k \rho^k Q_{k,k}(x, y|\rho, q),$$

and

$$(2.11) \quad q^{\binom{n}{2}} \rho^n (1-q)^n Q_{n,n}(x, y|\rho, q) = \sum_{k=0}^n (-1)^k q^{\binom{n-k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{\prod_{i=0}^{k-1} \omega\left(x\sqrt{1-q}/2, y\sqrt{1-q}/2|\rho q^i\right)}{(\rho^2)_{2k}}$$

with understanding that $\prod_{i=0}^{k-1}$ for $k=0$ is equal to 1.

Proof. Is shifted to section 4. □

Our main results follow in fact directly the results presented above.

Theorem 1. For $-1 < q \leq 1; x, y \in S(q); |\rho| < 1$: we have

i)

$$H_i(x|q) H_j(y|q) \frac{\prod_{i=0}^{\infty} \omega\left(x\sqrt{1-q}/2, y\sqrt{1-q}/2|\rho q^i\right)}{(\rho^2)_{\infty}} = \sum_{k=0}^{\infty} (-1)^k q^{\binom{k}{2}} \frac{\rho^k}{[k]_q!} Q_{i+k, j+k}(x, y|\rho, q).$$

In particular we get:

ii)

$$(2.12) \quad \frac{\prod_{i=0}^{\infty} \omega\left(x\sqrt{1-q}/2, y\sqrt{1-q}/2|\rho q^i\right)}{(\rho^2)_{\infty}} = \sum_{k=0}^{\infty} (-1)^k q^{\binom{k}{2}} \frac{\rho^k}{[k]_q!} Q_{k,k}(x, y|\rho, q).$$

3. OPEN PROBLEMS AND COMMENTS

Remark 2. The non-symmetric kernels constructed of bqH polynomials were given in [5]. Formula ii) of Proposition 2 gives its new interpretation. Besides, recall that these kernels were expressed using basic hypergeometric function ${}_3\phi_2$. Expansion on the left hand side of Proposition 2ii) gives new outlook on the properties of this function.

Notice also that for $q = 1$ we have $\eta(x|t, 1) = \exp(xt - \frac{t^2}{2})$, $H_n(x|t, 1) = H_n(x - t)$ and

$$\sum_{n \geq 0} \frac{\rho^n}{n!} H_n(x) H_n(y) = \exp\left(\frac{x^2}{2} - \frac{(x - \rho y)^2}{2(1 - \rho^2)}\right),$$

hence characteristic function of polynomials $Q_{i,j}$ can be calculated explicitly.

Similarly for $q = 0$ we have $\eta(x|t, 0) = \frac{1}{1-xt+t^2}$ (characteristic function of the Chebyshev polynomials) and $H_n(x|t, 0) = U_n(x/2) - tU_{n-1}(x/2)$ (see e.g. [10]) hence also in this case we can get explicit form of the characteristic function of polynomials $Q_{i,j}$.

Remark 3. First of all notice that the left hand side of (2.12) is equal to $1/\gamma_{0,0}(x, y|\rho, q) = f_N(x|q)/f_{CN}(x|y, \rho, q)$ and that it is a symmetric (with respect to x and y) function. In [8] there was presented (formula 5.3) an expansion of this function involving polynomials P_n and certain polynomials related to q -Hermite ones. The expansion was non-symmetric for every partial sum. Thus we get another expansion of known important special function.

Remark 4. Assertion i) of Proposition 2 states that polynomials $Q_{n,m}$ and $Q_{i,j}$ are orthogonal with respect to two dimensional measure μ_{2D} with the density given by (2.7) if only the $n + m \neq i + j$. Let us define space $\mathcal{L} = L_2(S^2(q), \mathcal{B}, \mu_{2D})$ of functions $f : S^2(q) \rightarrow \mathbb{R}$ square integrable with respect to the measure μ_{2D} . Do polynomials $Q_{m,n}$ constitute a base of this space? It seems that yes. We can define subspaces of $\Lambda_m = \text{span}\{Q_{m,0}, \dots, Q_{0,m}\}$ of polynomials that are linear combinations of polynomials $Q_{i,j}$ such that $i + j = m$. Subspaces Λ_m are mutually orthogonal. Besides following argument that polynomials are dense in \mathcal{L} we deduce that $\mathcal{L} = \bigoplus_{n=0}^{\infty} \Lambda_n$. What is the orthogonal base of \mathcal{L} ? We can calculate covariances between polynomials $Q_{i,j}$ from Λ_m following (2.5) and (1.17). Thus we can follow Gram-Schmidt orthogonalization procedure within the spaces Λ_m . Is the union of orthogonal bases of Λ_m an orthogonal base of \mathcal{L} ? Again it seems that yes. It would be interesting to find this base.

Remark 5. In 2001 Wünsche in [15] considered Hermite and Laguerre polynomials on the plane. He did not however relate his Hermite polynomials to any particular measure on the plane. In particular he defined Hermite polynomials depending on parameters forming a 2×2 matrix. This matrix is however not connected in any way to the covariance matrix of the measure with respect to which these polynomials are supposed to be orthogonal.

On the other hand definition of polynomials $Q_{i,j}$ depends heavily on the measure with the density f_{2D} . For $q = 1$ following (2.5), we have

$$Q_{i,j}(x, y|\rho, 1) = \sum_{k=0}^j (-1)^k \binom{j}{k} H_{j-k}(y) H_{k+i} \left(\frac{x - \rho y}{\sqrt{1 - \rho^2}} \right) / \left(\sqrt{1 - \rho^2} \right)^{k+i}.$$

Hence polynomials $Q_{i,j}(x, y, \rho, 1)$ are in fact another (different from that of Wünsche) family of two dimensional generalization of Hermite polynomials.

4. PROOFS

Proof of Proposition 1. First we will prove that

$$(4.1) \quad \sigma_n(\rho q^m|q) = \sigma_n(\rho q^{m-1}|q) - (1 - q)\rho q^{m-1}\sigma_{n+1}(\rho q^{m-1}|q).$$

We have:

$$\begin{aligned}\sigma_n(\rho|q) &= \sum_{i \geq 0} \frac{q^{mi} \rho^i}{[i]_q!} \xi_{n+i} = \sum_{i \geq 0} \frac{q^{(m-1)i} \rho^i}{[i]_q!} \xi_{n+i} - \sum_{i \geq 0} \frac{q^{(m-1)i} (1-q^i) \rho^i}{[i]_q!} \xi_{n+i} \\ &= \sigma_n(\rho q^{m-1}|q) - (1-q) \rho q^{m-1} \sum_{j \geq 0} \frac{q^{(m-1)j} t^j}{[j]_q!} \xi_{n+1+j}.\end{aligned}$$

Then we prove (1.19) by induction with respect to m . We see that it is true for $m = 1$. Hence let us assume that it is true for $m \leq k$. Let us consider $m = k + 1$. We have:

$$\begin{aligned}\sigma_n(\rho q^{k+1}|q) &= \sigma_n(\rho q^k|q) - (1-q) \rho q^k \sigma_{n+1}(\rho q^k|q) \\ &= \sum_{j=0}^k (-1)^j \begin{bmatrix} k \\ j \end{bmatrix}_q q^{\binom{j}{2}} (1-q)^j \rho^j \sigma_{n+j}(\rho|q) \\ &\quad - (1-q) \rho q^k \sum_{j=0}^k (-1)^j \begin{bmatrix} k \\ j \end{bmatrix}_q q^{\binom{j}{2}} (1-q)^j \rho^j \sigma_{n+1+j}(\rho|q) \\ &= \sigma_n(\rho|q) + (-1)^{k+1} \rho^{k+1} q^{\binom{k+1}{2}} + \sum_{j=1}^k (-1)^j \begin{bmatrix} k \\ j \end{bmatrix}_q q^{\binom{j}{2}} (1-q)^j \rho^j \sigma_{n+j}(\rho|q) \\ &\quad + \sum_{j=0}^{k-1} (-1)^{j+1} \begin{bmatrix} k \\ j \end{bmatrix}_q q^{k+\binom{j}{2}} (1-q)^{j+1} \rho^{j+1} \sigma_{n+j+1}(\rho|q).\end{aligned}$$

Now we change the index of summation from $j = s - 1$. and get:

$$\begin{aligned}\sigma_n(\rho q^k|q) - (1-q) \rho q^k \sigma_{n+1}(\rho q^k|q) \\ = \sigma_n(\rho|q) + (-1)^{k+1} \rho^{k+1} q^{\binom{k+1}{2}} + \sum_{j=1}^k (-1)^j \left(\begin{bmatrix} k \\ j \end{bmatrix}_q + q^{k-j+1} \begin{bmatrix} k \\ j-1 \end{bmatrix}_q \right) q^{\binom{j}{2}} (1-q)^j \rho^j \sigma_{n+j}(\rho|q)\end{aligned}$$

since $\binom{j-1}{2} + j - 1 = \binom{j}{2}$. Now we use the fact that $\begin{bmatrix} k \\ j \end{bmatrix}_q + q^{k-j+1} \begin{bmatrix} k \\ j-1 \end{bmatrix}_q = \begin{bmatrix} k+1 \\ j \end{bmatrix}_q$. Hence we see that (1.19) is true. \square

Proof of Proposition 2. i) We use (2.5), assume that $i > m$. We have:

$$\begin{aligned}& \int_{S^2(q)} Q_{i,j}(x, y|q) Q_{m,k}(x, y|\rho, q) f_{2D}(x, y|\rho, q) dx dy = \\ & \sum_{s=0}^j \sum_{t=0}^k (-1)^{s+t} q^{\binom{s}{2}} q^{\binom{t}{2}} \begin{bmatrix} j \\ s \end{bmatrix}_q \begin{bmatrix} k \\ t \end{bmatrix}_q \rho^{s+t} \frac{1}{(\rho^2)_{i+s} (\rho^2)_{m+t}} \int_{S(q)} H_{j-s}(y) H_{k-t}(y|q) f_N(y|q) \\ & \quad \times \int_{S(q)} P_{i+s}(x|y, \rho, q) P_{m+t}(x|y, \rho, q) f_{CN}(x|y, \rho, q) dx dy = \\ & \quad (-1)^{i-m} \rho^{i-m} \sum_{s=0 \vee m-i}^{j \wedge k+m-i} q^{\binom{s}{2} + \binom{i-m+s}{2}} \begin{bmatrix} j \\ s \end{bmatrix}_q \begin{bmatrix} k \\ i+s-m \end{bmatrix}_q \times \\ & \quad \rho^{2s} \frac{[i+s]_q!}{(\rho^2)_{i+s}} \int_{S(q)} H_{j-s}(y) H_{k+m-i-s}(y|q) f_N(y|q) dy = 0\end{aligned}$$

if $j - s \neq k + m - i - s$ i.e. if $j + i \neq k + m$.

ii) We have

$$\begin{aligned} \sum_{i \geq 0, j \geq 0} \frac{s^i t^j}{[i]_q! [j]_q!} Q_{i,j}(x, y | \rho, q) &= \frac{1}{\gamma_{0,0}(x, y, \rho, q)} \sum_{i \geq 0, j \geq 0} \frac{t^i s^j}{[i]_q! [j]_q!} \sum_{n \geq 0} \frac{\rho^n}{[n]_q!} H_{i+n}(x|q) H_{n+j}(y|q) \\ &= \sum_{n \geq 0} \frac{\rho^n}{[n]_q!} \sum_{j=0}^{\infty} \frac{s^j}{[j]_q!} H_{n+j}(y|q) \sum_{i \geq 0} \frac{t^i}{[i]_q!} H_{n+i}(x|q). \end{aligned}$$

Now we use assertion i) of Lemma 1 and get

$$\sum_{i \geq 0, j \geq 0} \frac{s^i t^j}{[i]_q! [j]_q!} Q_{i,j}(x, y | \rho, q) = \frac{1}{\prod_{i=0}^{\infty} V(x|tq^i, q)} \sum_{n \geq 0} \frac{\rho^n}{[n]_q!} H_n(x|t, q) \sum_{j=0}^{\infty} \frac{s^j}{[j]_q!} H_{n+j}(y|q).$$

Arguing in the similar way in the case if the series $\sum_{j=0}^{\infty} \frac{s^j}{[j]_q!} H_{n+j}(y|q)$, we get final result.

iii) First we notice that from (2.4) it follows that for $x, y \in S(q); \rho^2 < 1, -1 < q \leq 1$:

$$\begin{aligned} \gamma_{i,j}(x, y, \rho q^m, q) &= Q_{i,j}(x, y, \rho q^m, q) \frac{(\rho^2 q^{2m})_{\infty}}{\prod_{i=0}^{\infty} \omega(x\sqrt{1-q}/2, y\sqrt{1-q}/2 | \rho q^{m+i})} \\ &= Q_{i,j}(x, y, \rho q^m, q) \frac{\prod_{i=0}^{m-1} \omega(x\sqrt{1-q}/2, y\sqrt{1-q}/2 | \rho q^i)}{(\rho^2)_{2m}} \gamma_{0,0}(x, y, \rho, q), \end{aligned}$$

and also that $\gamma_{0,0}(x, y, \rho, q) = \frac{(\rho^2)_{\infty}}{\prod_{i=0}^{\infty} \omega(x\sqrt{1-q}/2, y\sqrt{1-q}/2 | \rho q^i)}$. Then we apply (2.1) to $\gamma_{i,j}$ above and then use (1.19) and cancel out $\gamma_{0,0}$ on both sides of (2.2). Finally we observe that on both sides we have polynomials hence one can extend the identity for all values of the variables. To get other formula of this assertion we argue by induction checking that the equality is true for $n = 0$. Then we put (2.10) into (2.11) and get:

$$\begin{aligned} \sum_{k=0}^n (-1)^k q^{\binom{n-k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{\prod_{i=0}^{k-1} \omega(x\sqrt{1-q}/2, y\sqrt{1-q}/2 | \rho q^i)}{(\rho^2)_{2k}} &= \\ \sum_{k=0}^n (-1)^k q^{\binom{n-k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \sum_{j=0}^k (-1)^j \begin{bmatrix} k \\ j \end{bmatrix}_q q^{\binom{j}{2}} \rho^j Q_{j,j}(x, y | \rho, q) &= \\ \sum_{j=0}^n (-1)^j q^{\binom{j}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_q \rho^j Q_{j,j}(x, y | \rho, q) \sum_{k=j}^n (-1)^k q^{\binom{n-k}{2}} \begin{bmatrix} n-j \\ k-j \end{bmatrix}_q &= \\ \sum_{j=0}^n (-1)^j q^{\binom{j}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_q \rho^j Q_{j,j}(x, y | \rho, q) \sum_{m=0}^{n-j} (-1)^{m+j} q^{\binom{m}{2}} \begin{bmatrix} n-j \\ m \end{bmatrix}_q &= \\ q^{\binom{n}{2}} \rho^n (1-q)^n Q_{n,n}(x, y | \rho, q) & \end{aligned}$$

since $\forall n \geq 1 : \sum_{i=0}^n (-1)^i q^{\binom{i}{2}} \begin{bmatrix} n \\ i \end{bmatrix}_q = 0$. □

Proof of Theorem 1. To get i) we pass in (2.8) with m to infinity noting by (2.5) and (1.16) that $Q_{n,m}(x, y | 0, q) = H_n(x|q) H_m(y|q)$. On the way we observe that $\lim_{m \rightarrow \infty} \begin{bmatrix} m \\ k \end{bmatrix}_q = \frac{1}{(q)_k} = (1-q)^{-k} \frac{1}{[k]_q!}$. □

Proof of Lemma . We have

$$\begin{aligned} x\eta_n(x, t|q) &= \sum_{i \geq 0} \frac{t^i}{[i]_q!} (H_{n+1+i}(x|q) + [n+i]_q H_{n-1+i}(x|q)) = \\ \eta_{n+1}(x, t|q) &+ \sum_{i \geq 0} \frac{t^i}{[i]_q!} [i]_q H_{n-1+i}(x|q) + [n]_q \sum_{i \geq 0} \frac{q^i t^i}{[i]_q!} H_{n-1+i}(x|q) = \\ &\eta_{n+1}(x, t|q) + t\eta_n(x, t|q) + [n]_q \eta_{n-1}(x, qt|q). \end{aligned}$$

But we have (1.19). Hence applying it for $m = 1$ we get.

$$\begin{aligned} x\eta_n(x, t|q) &= \eta_{n+1}(x, t|q) + t\eta_n(x, t|q) + [n]_q \eta_{n-1}(x, qt|q) = \\ \eta_{n+1}(x, t|q) &+ t\eta_n(x, t|q) + [n]_q \eta_{n-1}(x, t|q) - [n]_q (1-q)t\eta_n(x, t|q) = \\ \eta_{n+1}(x, t|q) &+ t\eta_n(x, t|q) + [n]_q \eta_{n-1}(x, t|q) - (1-q^n)t\eta_n(x, t|q) = \\ &\eta_{n+1}(x, t|q) + tq^n \eta_n(x, t|q) + [n]_q \eta_{n-1}(x, t|q) \end{aligned}$$

$$(4.2) \quad (x - tq^n)\eta_n(x, t|q) = \eta_{n+1}(x, t|q) + [n]_q \eta_{n-1}(x, t|q).$$

We have also $\eta_0(x, t|q) = \frac{1}{\prod_{i=0}^{\infty} v(x|tq^i, q)}$, where $v(x|t, q)$ is given by (1.8), $\eta_1(x, t|q) = \sum_{i \geq 0} \frac{t^i}{[i]_q!} (xH_i(x|q) - [i]_q H_{i-1}(x|q)) = x\eta_0(x, t|q) - t\eta_0(x, t|q) = (x - t)\eta_0(x, t|q)$. One can easily notice that (4.2) is exactly the same as the 3-term recurrence of so called big Hermite polynomials i.e. (1.14). So we deduce that

$$\eta_n(x, t|q) = H_n(x|t, q) \eta_0(x, t|q).$$

□

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